

บทความวิจัย (Research Article)

Relationship between magic fractions and decimals based on Lucas and Fibonacci numbers

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Received: 13/03/2026r; Revised: 10/04/2026; Accepted: 16/04/2026

KEYWORDS

Fibonacci numbers
Lucas numbers
fractions
decimals

ABSTRACT

This research aims to study the relationship between fractions and decimals derived from the Lucas and Fibonacci sequences by considering the decimal values of fractions with digit numbers in the Lucas and Fibonacci sequences. We analyze the pattern of repeating decimals and their mathematical properties. The results show a distinct correlation between Lucas and Fibonacci sequences in the occurrence of the decimals. We give the exact count of Fibonacci numbers in decimal forms of remarkable fractions studied by James Smoak (2003). These theorems offer potential applications in numerical analysis, particularly regarding approximations and error estimation.

1. Introduction

Fibonacci numbers are famous due to their interesting properties, including their connection to the Golden Ratio and their appearance in various natural patterns. Lucas numbers are related to Fibonacci numbers because they follow the same recursive relationship. Fibonacci and Lucas numbers play a critical role in analyzing the performance of the Euclidean algorithm, which is used to find the greatest common factor of two integers. The Euclidean algorithm performs slowest if the input data are two consecutive Fibonacci numbers. Furthermore, ratios of Fibonacci and Lucas numbers have been applied to modified fraction expansions such as the Engel and Pierce expansions (Knopfmacher and Mays, 1995). Regarding the representation of natural numbers, Zeckendorf's Theorem (Keller, 1972) states that every natural number can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers. From this important theorem, many further research studies have been developed, such as Zeckendorf games (Baird-Smith et al., 2018; Cusenza et al., 2021), a pooling layer in convolutional neural network (Vigneron et al., 2021), and a probabilistic approach for Markov chains (Ben-Ari and Miller, 2016). Additionally, in the field of image encryption, several researchers (Batool and Waseem, 2019; Maiti et al., 2023) have developed an encryption method using Fibonacci and Lucas numbers. These numbers are continued being researched, and their properties are used in many fields (see (Debnath, 2011; Vajda, 2008; Koshy, 2019) for more details).

In this study, we are interested in research article by Smoak and Osler (Smoak and Osler, 2003). They studied the following fascinating fractions related to Fibonacci numbers:

$$\frac{100}{89} = 1.1235955056 \dots$$

$$\frac{10000}{9899} = 1.0102030508132134559046368 \dots$$

$$\frac{1000000}{998999} = 1.001\ 002\ 003\ 005\ 008\ 013\ 021\ 034\ 055\ 089\ 144\ 233\ 377\ 610\ 98859958818777596 \dots$$

From the three examples above, it is evident that the decimal expansions of these fractions contain Fibonacci numbers. The decimal expansion of the first fraction produces the first five Fibonacci numbers (starting with $F_1 = 1$) before transitioning into other digits. The decimal expansion of the second fraction contains the first ten Fibonacci numbers, while the decimal expansion of the third fraction contains the first fifteen Fibonacci numbers. These three fractions have similar structure. The numerator of the fractions is in the form of 10^{2n} and the denominator of the fractions is in the form of $10^{2n} - 10^n - 1$ for integers $n = 1, 2, 3$. Smoak and Osler proved that this pattern continues indefinitely for all integers n .

This research determines the exact number of Fibonacci numbers appearing in the decimal expansions of the family of fractions above. Furthermore, we investigate two additional families of fractions whose decimal expansions generate Lucas numbers. Representative members of the first family of fractions generating Lucas numbers are shown as follows

$$\frac{12}{89} = 0.1\ 3\ 4\ 83146 \dots$$

$$\frac{102}{9899} = 0.01\ 03\ 04\ 07\ 11\ 18\ 29\ 47\ 772502 \dots$$

$$\frac{1002}{998999} = 0.001\ 003\ 004\ 007\ 011\ 018\ 029\ 047\ 076\ 123\ 199\ 322\ 521\ 84436621057 \dots$$

From the three examples above, it is observed that the decimal expansions of these fractions contain Lucas numbers. The decimal expansion of the first fraction displays the first three Lucas numbers (starting with $L_1 = 1$) before transitioning into other digits. The decimal expansion of the second fraction contains the first eight Lucas numbers, while the decimal expansion of the third fraction contains the first thirteen Lucas numbers. These three fractions have a similar structure. The numerator of the fractions is in the form of $10^n + 2$ and the denominator of the fractions is in the form of $10^{2n} - 10^n - 1$ for integers $n = 1, 2, 3$.

The examples of the member of another family of fractions generating Lucas numbers are shown as follows

$$\frac{19}{89} = 0.2\ 1\ 3\ 4\ 83146 \dots$$

$$\frac{199}{9899} = 0.02\ 01\ 03\ 04\ 07\ 11\ 18\ 29\ 47\ 7725022 \dots$$

$$\frac{1999}{998999} = 0.002\ 001\ 003\ 004\ 007\ 011\ 018\ 029\ 047\ 076\ 123\ 199\ 322\ 521\ 84436621 \dots$$

From the three examples above, it is evident that the decimal expansions of the fractions contain Lucas numbers. The decimal expansion of the first fraction produces the first four Lucas numbers (starting with $L_0 = 2$) before being followed by other digits. The decimal expansion of the second fraction contains the first nine Lucas numbers, while the decimal expansion of the third fraction contains the first fourteen Lucas numbers. These three fractions have a similar structure. The numerator of the fractions is in the form of $2(10^n) - 1$ and the denominator of the fractions is in the form of $10^{2n} - 10^n - 1$ for integers $n = 1, 2, 3$.

In the Results section, we prove that the patterns of these decimal expansions of the fractions of these two families generating Lucas numbers hold true for all positive integers. In addition, we provide the exact count of Fibonacci and Lucas numbers appearing in decimal expansions of all three families of fractions.

2. Preliminary

In this section, we provide the fundamental definitions and notation used throughout this study. Additionally, we review essential properties of the Fibonacci and Lucas sequences that are necessary for proving the main results of this research.

2.1 Fibonacci numbers

The **Fibonacci numbers** are defined by the recurrence relation:

$$F_{n+1} = F_n + F_{n-1}$$

for all positive integers $n \geq 2$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. This recurrence relation generates the **Fibonacci sequence**, denoted by $\{F_n\}$. The first 15 Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, and 377. The n^{th} Fibonacci number can be calculated using Binet's formula as follows:

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$

for all nonnegative integers n , where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

2.2 Lucas numbers

The **Lucas numbers** are defined by the recurrence relation:

$$L_{n+1} = L_n + L_{n-1}$$

for all positive integers $n \geq 2$ with the initial conditions $L_0 = 2$ and $L_1 = 1$. This recurrence relation generates the **Lucas sequence**, denoted by $\{L_n\}$. The first 15 Lucas numbers are 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, and 843. The n^{th} Lucas number can be calculated using Binet's formula as follows: for all nonnegative integers n ,

$$L_n = \phi^n + (1-\phi)^n = \phi^n + (-\phi)^{-n}$$

for all nonnegative integers n , where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Fibonacci and Lucas number are connected by various algebraic identities. For example,

- $L_n = F_{n-1} + F_{n+1} = 2F_{n+1} + F_n$
- $F_n = \frac{L_{n-1} + L_{n+1}}{5}$

2.3 Floor function

The **floor function**, denoted by $[x]$, maps a real number x to the greatest integer less than or equal to x . For example,

- $[0.4] = 0$
- $[1.6] = 1$
- $[-0.4] = -1$
- $[-1.6] = -2$

The main properties applied in this research are as follows: for any real number x , there exists a unique integer m that satisfying the following statements:

- $[x] = m$ if and only if $m \leq x < m + 1$;
- $[x] = m$ if and only if $x - 1 < m \leq x$.

3. Results

This section presents the proofs for the properties of the three families of fractions, starting with *Lemma 1* and *Lemma 2*.

The following lemma establishes the relationship between linear recurrence relations and their corresponding power series.

Lemma 1. Let a_n be a sequence of real numbers generated by the recurrence relation $a_n = a_{n-1} + a_{n-2}$ for all positive integers $n \geq 2$ with initial condition that a_0, a_1 as constants. Then, the generating function for the sequence is given by

$$\sum_{n=0}^{\infty} a_n x^n = \frac{a_0 + (a_1 - a_0)x}{1 - x - x^2}$$

Proof. Consider the following equations

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n (1 - x - x^2) &= \sum_{n=0}^{\infty} a_n x^n - x \sum_{n=0}^{\infty} a_n x^n - x^2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n \\ &= a_0 + a_1 x - a_0 x + \sum_{n=2}^{\infty} (a_n - a_{n-1} - a_{n-2}) x^n. \end{aligned}$$

From the recurrence relation, it follows that $a_n - a_{n-1} - a_{n-2} = 0$ for all positive integers $n \geq 2$ and

$$\sum_{n=0}^{\infty} a_n x^n (1 - x - x^2) = a_0 + a_1 x - a_0 x.$$

Therefore,

$$\frac{a_0 + (a_1 - a_0)x}{1 - x - x^2} = \sum_{n=0}^{\infty} a_n x^n$$

□

The examples of *Lemma 1* are Fibonacci and Lucas sequences as follows:

$$\begin{aligned} \frac{1}{1 - x - x^2} &= \frac{F_1 + (F_2 - F_1)x}{1 - x - x^2} = F_1 + F_2 x + \cdots + F_n x^{n-1} + \cdots \\ \frac{2 - x}{1 - x - x^2} &= \frac{L_0 + (L_1 - L_0)x}{1 - x - x^2} = L_0 + L_1 x + L_2 x^2 + \cdots + L_n x^n + \cdots \end{aligned}$$

where $F_1 = F_2 = L_1 = 1$ and $L_0 = 2$. Next, we establish *Lemma 2* that shows the relationship between Lucas numbers and the golden ratio.

Lemma 2. The number of digits in the n^{th} Lucas number is equal to $1 + \lfloor n \log \phi \rfloor$.

Proof. Let n be a nonnegative integer. There exists a unique N such that the n^{th} Lucas number L_n can be expressed as

$$d_N d_{N-1} \dots d_2 d_1$$

where d_i denotes the digit associated with the 10^{i-1} power in the decimal expansion for all positive integers i . Therefore, L_n can also be written as $L_n = (d_N \cdot d_{N-1} \dots d_2 d_1) 10^{N-1}$.

It follows that

$$\begin{aligned} \log L_n &= \log(d_N \cdot d_{N-1} \dots d_2 d_1) + \log(10)^{N-1} \\ \log L_n &= \log(d_N \cdot d_{N-1} \dots d_2 d_1) + (N - 1) \log 10 \\ \log L_n &= y + N - 1 \text{ where } y = \log(d_N \cdot d_{N-1} \dots d_2 d_1). \end{aligned}$$

We have $N = \log L_n - y + 1$. Since $0 = \log(1) \leq y = \log(d_N \cdot d_{N-1} \dots d_2 d_1) \leq \log(10) = 1$, $N = \lfloor \log L_n \rfloor + 1$. Note that N is the number of digits in the n^{th} Lucas number.

From Binet's formula $L_n = \phi^n + (1 - \phi)^n = \phi^n + (-\phi)^{-n}$, we obtain $\log(L_n) = \log(\phi^n + (1 - \phi)^n) = \log\left(\phi^n + \frac{(-1)^n}{\phi^n}\right)$. Thus $\log\left(\phi^n - \frac{1}{\phi^n}\right) \leq \log(L_n) \leq \log\left(\phi^n + \frac{1}{\phi^n}\right)$.

Consider the $f(n) = \log\left(1 - \frac{1}{\phi^{2n}}\right)$. Since $f(n)$ is strictly increasing, it follows that $-0.00355 < f(9) < f(n)$ for all $n > 9$. Conversely, let $g(n) = \log\left(1 + \frac{1}{\phi^{2n}}\right)$. Since $g(n)$ is strictly decreasing, it follows that $0.00355 > g(9) > g(n)$ for all $n > 9$.

Note that $\log\left(\phi^n - \frac{1}{\phi^n}\right) = \log\left(\phi^n \left(1 - \frac{1}{\phi^{2n}}\right)\right) = \log(\phi^n) + \log\left(1 - \frac{1}{\phi^{2n}}\right) > \log(\phi^n) - 0.00355$ for all integers $n > 9$. Similarly, note that $\log\left(\phi^n + \frac{1}{\phi^n}\right) = \log\left(\phi^n \left(1 + \frac{1}{\phi^{2n}}\right)\right) = \log(\phi^n) + \log\left(1 + \frac{1}{\phi^{2n}}\right) < \log(\phi^n) + 0.00355$ for all integers $n > 9$. Thus, $\log \phi^n - 0.00355 < \log(L_n) < \log \phi^n + 0.00355$ for all integers $n > 9$.

If $\log \phi^n - 0.00355 - y < \log L_n - y = \lfloor \log L_n \rfloor$, then $\log \phi^n - 0.00355 < \lfloor \log L_n \rfloor + y = \lfloor \log L_n \rfloor$. We have $\lfloor \log \phi^n \rfloor - 1 < \lfloor \log L_n \rfloor$ for all integers $n > 9$. If $\lfloor \log L_n \rfloor = \log L_n - y < \log \phi^n + 0.00355 - y$, then $\lfloor \log L_n \rfloor = \lfloor \log L_n \rfloor + y < \log \phi^n + 0.00355$. We have $\lfloor \log L_n \rfloor < \lfloor \log \phi^n \rfloor + 1$ for all integers $n > 9$. It can be concluded that $\lfloor \log \phi^n \rfloor - 1 < \lfloor \log L_n \rfloor < \lfloor \log \phi^n \rfloor + 1$ for all integers $n > 9$. In addition, one can calculate to verify $\lfloor \log L_n \rfloor = \lfloor \log \phi^n \rfloor = 0$ for all integers $n \in \{0, 1, 2, 3, 4\}$ and $\lfloor \log L_n \rfloor = \lfloor \log \phi^n \rfloor = 1$ for all integers $n \in \{5, 6, 7, 8, 9\}$.

Therefore, $\lfloor \log L_n \rfloor = \lfloor \log \phi^n \rfloor = \lfloor n \log \phi \rfloor$ for all nonnegative integers n .

Since $N = \lfloor \log L_n \rfloor + 1$ and $\lfloor \log L_n \rfloor = \lfloor n \log \phi \rfloor$, $N = \lfloor n \log \phi \rfloor + 1$. This proves that the number of digits in the n^{th} Lucas number is equal to $1 + \lfloor n \log \phi \rfloor$. □

The examples of Lemma 2 are shown below:

- The number of digits in $L_0 = 2$ is equal to $1 + \lfloor 0(\log \phi) \rfloor = 1$;
- The number of digits in $L_1 = 1$ is equal to $1 + \lfloor 1(\log \phi) \rfloor = 1$;
- The number of digits in $L_5 = 11$ is equal to $1 + \lfloor 5(\log \phi) \rfloor = 2$;
- The number of digits in $L_{15} = 1364$ is equal to $1 + \lfloor 15(\log \phi) \rfloor = 4$;
- The number of digits in $L_{45} = 2537720636$ is equal to $1 + \lfloor 45(\log \phi) \rfloor = 10$;

Next, we examine the two main theorems stating the properties of the families of magic fractions.

Theorem 1. Let a sequence $\{R_n\}_{n=1}^{\infty}$ be a family of fractions where $R_n = \frac{2(10^n)-1}{10^{2n}-10^{n-1}}$ for all positive integers n . Then the decimal expansion of the fractions R_n generates the first M Lucas numbers starting with $L_0 = 2$ where $M = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi \rfloor = n\}$.

Proof. From Lemma 1 and the recurrence relation of Lucas sequence $L_n = L_{n-1} + L_{n-2}$ for all positive integers $n > 1$ where $L_1 = 1$ and $L_0 = 2$, we have

$$L_0 + L_1x + L_2x^2 + \dots = \frac{2-x}{1-x-x^2}. \quad (1)$$

Multiplying both sides of Equation (1) by x , we have

$$L_0x + L_1x^2 + L_2x^3 + \dots = \frac{2x-x^2}{1-x-x^2}. \quad (2)$$

Replacing $x = 10^{-n}$ in Equation (2), we have

$$L_0(10^{-n}) + L_1(10^{-2n}) + L_2(10^{-3n}) + \dots = \frac{2(10^{-n}) - 10^{-2n}}{1 - 10^{-n} - 10^{-2n}}. \quad (3)$$

The right-hand side of Equation (3) gives the family of fractions when n is positive integer as follows:

$$\left\{ \frac{2(0.1) - (0.01)}{1 - (0.1) - (0.01)}, \frac{2(0.01) - (0.0001)}{1 - (0.01) - (0.0001)}, \frac{2(0.001) - (0.000001)}{1 - (0.001) - (0.000001)}, \dots \right\}.$$

That is,

$$\left\{ \frac{19}{89}, \frac{199}{9899}, \frac{1999}{998999}, \dots \right\}.$$

Hence, the right-hand side of Equation (3) provides the family $\{R_n\}_{n=1}^{\infty}$, while the left-hand side of Equation (3) gives the decimal expansion of the fractions R_n for all positive integers n . For example,

$$R_1 = \frac{19}{89} = L_0(0.1) + L_1(0.01) + L_2(0.001) + \dots = 0.2 + 0.01 + 0.003 + 0.004 + \dots;$$

$$R_2 = \frac{199}{9899} = L_0(0.01) + L_1(0.0001) + L_2(0.000001) + \dots = 0.02 + 0.0001 + 0.000003 + \dots.$$

Thus, each R_n generates the decimal containing the first M Lucas numbers for some positive integer M . Since some Lucas numbers have a place value exceeding 10^n , carrying over may occur; consequently, the summation on the left-hand side of Equation (3) does not display every Lucas number.

From Lemma 2, the number of digits in the n^{th} Lucas number is equal to $1 + \lfloor n \log \phi \rfloor$. Then, L_M is the greatest Lucas number with its place value less than 10^{n+1} , where $M = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi \rfloor = n\}$. So, L_{M+1} is the smallest Lucas number with its place value equal to 10^{n+1} and L_M is the first Lucas number that fails to appear clearly in the decimal expansion of the fraction due to the carrying effect from subsequent terms.

It can be concluded that the decimal expansion of the fractions R_n generates the first M Lucas numbers starting with $L_0 = 2$ where $M = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi \rfloor = n\}$. □

From Theorem 1, we can see that, for example, $R_1 = \frac{19}{89}$ generates the first four Lucas numbers starting with $L_0 = 2$ where $4 = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi \rfloor = 1\} = \max\{0, 1, 2, 3, 4\}$. That is,

$$R_1 = \frac{19}{89} = 0.2 + 0.01 + 0.003 + 0.0004 + 0.00007 + 0.000011 + \dots = 0.213483 \dots$$

which shows the only first four Lucas numbers: 2, 1, 3, and 4.

Next, we prove another main theorem about the family of magic fractions generating Lucas numbers starting with $L_1 = 1$.

Theorem 2. Let a sequence $\{P_n\}_{n=1}^{\infty}$ be a family of fractions where $P_n = \frac{10^{n+2}}{10^{2n-1} - 1}$ for all positive integers n . Then the decimal expansion of the fractions P_n generates the first $M - 1$ Lucas numbers starting with $L_1 = 1$ where $M = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi \rfloor = n\}$.

Proof. From Lemma 1 and the recurrence relation of Lucas sequence $L_n = L_{n-1} + L_{n-2}$ for all positive integers $n > 2$ where $L_2 = 3$ and $L_1 = 1$, we have

$$L_1 + L_2x + L_3x^2 + \dots = \frac{1 + 2x}{1 - x - x^2}. \quad (5)$$

Multiplying both sides of Equation (1) by x , we have

$$L_1x + L_2x^2 + L_3x^3 + \dots = \frac{x + 2x^2}{1 - x - x^2}. \quad (6)$$

Replacing $x = 10^{-n}$ in Equation (2), we have

$$L_1(10^{-n}) + L_2(10^{-2n}) + L_3(10^{-3n}) + \dots = \frac{10^{-n} + 2(10^{-2n})}{1 - 10^{-n} - 10^{-2n}}. \quad (7)$$

The right-hand side of Equation (7) gives the family of fractions when n is positive integer as follows:

$$\left\{ \frac{0.1 + 2(0.01)}{1 - (0.1) - (0.01)}, \quad \frac{0.01 + 2(0.0001)}{1 - (0.01) - (0.0001)}, \quad \frac{0.001 + 2(0.000001)}{1 - (0.001) - (0.000001)}, \quad \dots \right\}.$$

That is,

$$\left\{ \frac{12}{89}, \quad \frac{102}{9899}, \quad \frac{1002}{998999}, \quad \dots \right\}.$$

Hence, the right-hand side of Equation (7) gives the family $\{P_n\}_{n=1}^{\infty}$, while the left-hand side of Equation (7) gives the decimal expansion of the fractions P_n for all positive integers n . For example,

$$P_1 = \frac{12}{89} = L_1(0.1) + L_2(0.01) + L_3(0.001) + \dots = 0.1 + 0.03 + 0.004 + 0.007 + \dots;$$

$$P_2 = \frac{102}{9899} = L_1(0.01) + L_2(0.0001) + L_3(0.000001) + \dots = 0.01 + 0.0003 + 0.000004 + \dots.$$

Thus, each P_n generates the decimal containing the first N Lucas numbers for some positive integer N . Since some Lucas numbers have a place value exceeding 10^n , carrying over may occur; consequently, the summation on the left-hand side of Equation (7) does not display every Lucas number.

From Lemma 2, the number of digits in the n^{th} Lucas number is equal to $1 + \lfloor n \log \phi \rfloor$. Then, L_M is the greatest Lucas number with its place value less than 10^{n+1} , where $M = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi \rfloor = n\}$. So, L_{M+1} is the smallest Lucas number with its place value equal to 10^{n+1} and L_M is the first Lucas number that fails to appear clearly in the decimal expansion of the fraction due to the carrying effect from subsequent terms. It can be concluded that the decimal expansion of the fractions P_n generates the first $M - 1$ Lucas numbers starting with $L_1 = 1$ where $M = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi \rfloor = n\}$.

□

From Theorem 2, we can see that, for example, $P_1 = \frac{12}{89}$ generates the first three Lucas numbers starting with $L_1 = 1$ where $4 = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi \rfloor = 1\} = \max\{0, 1, 2, 3, 4\}$. That is,

$$P_1 = \frac{12}{89} = 0.1 + 0.03 + 0.004 + 0.0007 + 0.00011 + \dots = 0.13483 \dots$$

which shows the only first three Lucas numbers: 1, 3, and 4.

Based on James Smoak's study in 2003, that the n^{th} Fibonacci number F_n can be written as

$$d_N d_{N-1} \dots d_2 d_1$$

where d_i represents a digit in the 10^{i-1} s place. Therefore, F_n can also be written as $F_n = (d_N \cdot d_{N-1} \dots d_2 d_1) 10^{N-1}$. It follows that $N = \log F_n - y + 1$ where $y = \log(d_N \cdot d_{N-1} \dots d_2 d_1)$ and N is the number of digits in the n^{th} Fibonacci number. Since $F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$ when $\phi = \frac{1+\sqrt{5}}{2}$, we have $N = \left\lfloor \log \frac{\phi^n}{\sqrt{5}} \right\rfloor + 1 = \lfloor n \log \phi - \log \sqrt{5} \rfloor + 1$. Therefore, we obtain the following corollary.

Corollary 3. Let a sequence $\{E_n\}_{n=1}^\infty$ be a family of fractions where $E_n = \frac{10^n}{10^{2n}-10^{n-1}}$ for all positive integers n . Then the decimal expansion of the fractions E_n generates the first $M - 1$ Fibonacci numbers starting with $F_1 = 1$ where $M = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi - \log \sqrt{5} \rfloor = n\}$.

From Corollary 3, we can see that, for example, $E_1 = \frac{10}{89}$ generates the first five Fibonacci numbers starting with $F_1 = 1$ where $6 = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi - \log \sqrt{5} \rfloor = 1\} = \max\{0, 1, 2, 3, 4, 5, 6\}$. That is,

$$E_1 = \frac{10}{89} = 0.1 + 0.01 + 0.002 + 0.0003 + 0.00005 + 0.000008 + \dots = 0.11235955 \dots$$

which displays the only first five Fibonacci numbers: 1, 1, 2, 3 and 5. For another example, $E_2 = \frac{10}{9899}$ generates the first ten Fibonacci numbers starting with $F_1 = 1$ where $11 = \max_{m \in \mathbb{N}} \{1 + \lfloor m \log \phi - \log \sqrt{5} \rfloor = 2\} = \max\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. That is,

$$E_2 = \frac{100}{9899} = 0.01 + 0.0001 + 0.000002 + 0.00000003 + \dots = 0.010102030508132134559046 \dots$$

which shows the only first ten Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, and 55.

We can notice that the three families of magic fractions $\{R_n\}_{n=1}^\infty$, $\{P_n\}_{n=1}^\infty$, and $\{E_n\}_{n=1}^\infty$ in Theorem 1, Theorem 2, and Corollary 3, respectively, are related. Since $\{R_n\}_{n=1}^\infty$ and $\{P_n\}_{n=1}^\infty$ are families of fractions generating Lucas numbers and $\{E_n\}_{n=1}^\infty$ is a family of fractions generating Fibonacci numbers,

$$2P_n - R_n = \frac{5}{10^n} E_n$$

for all positive integers n .

4. Conclusions

This research analyzes the pattern of repeated decimals of magic fractions and their special properties. There are three families that are studied as follows:

$$\{R_n\}_{n=1}^{\infty} = \left\{ \frac{19}{89}, \frac{199}{9899}, \frac{1999}{998999}, \dots, \frac{2(10^n) - 1}{\underbrace{10^{2n} - 10^n - 1}_{R_n}}, \dots \right\}$$

$$\{P_n\}_{n=1}^{\infty} = \left\{ \frac{12}{89}, \frac{102}{9899}, \frac{1002}{998999}, \dots, \frac{10^n + 2}{\underbrace{10^{2n} - 10^n - 1}_{P_n}}, \dots \right\}$$

$$\{E_n\}_{n=1}^{\infty} = \left\{ \frac{10}{89}, \frac{100}{9899}, \frac{1000}{998999}, \dots, \frac{10^n}{\underbrace{10^{2n} - 10^n - 1}_{E_n}}, \dots \right\}$$

The fractions in $\{R_n\}_{n=1}^{\infty}$ and $\{P_n\}_{n=1}^{\infty}$ generate Lucas numbers and the fractions in $\{E_n\}_{n=1}^{\infty}$ generate Fibonacci numbers. We give the exact number of Fibonacci and Lucas numbers in decimal forms of remarkable fractions. Moreover, we find the relation of these three families that

$$2P_n - R_n = \frac{5}{10^n} E_n$$

for all positive integers n . Our theorems may be applied in numerical analysis such as approximations and errors.

Acknowledgements

We acknowledge the support of Division of Mathematics and Statistics, Department of Computational Science and Digital Technology for providing the space and infrastructure that supported this research.

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